"Nonlinear" radiation of a sine-Gordon soliton generated by a constant external field

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Cooperation of the nonlinearity and of a constant external field below a critical value $f_{\rm crit}$ is shown to generate a traveling radiation in a forced sine-Gordon system.

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I. INTRODUCTION

There exists a general approach to find solutions of the nonlinear Klein-Gordon (KG) class: stable stationary or traveling-soliton solutions to the nonlinear wave equation and small oscillations (linear excitations) about these solutions [1]. In general, this scheme applies both for quantum and classical cases; while in the former case it represents the WKB approximation to the solution, in the latter one it tests the stability of the static or of the traveling solution against external perturbations. The linear perturbation scheme for the oscillations is obtained by linearization of the KG equations about the soliton or the ground-state solutions and it also represents a basis for the linear-stability analysis of the respective soliton solutions. However, in the presence of constant external perturbations this linear-perturbation scheme can become insufficient. Namely, the higher- (third-) order terms in the perturbation expansion in cooperation with a constant external field imply new stable traveling oscillations with the amplitude proportional to the field. This kind of solution is of the family of the driven traveling solutions for a ϕ^4 system obtained by Lal [2]. The driven ϕ^4 problem as a short-range approximation of a sine-Gordon one in a low constant field includes its relevant nonlinear properties. Here, the nonlinearity mediates the transformation of the excess energy gained by the system from the field to the emitted radiation. The significant feature of the oscillation is the proportionality of the respective amplitude to the external field. Numerical solution of the constantly driven sine-Gordon equation, namely the dynamics of the antisoliton [3] or of a soliton (this paper) for weak fields $f<f_{\rm crit}$, exhibits oscillations of the constant part of the kink with the amplitude of the oscillations proportional to the field as well as generation of traveling oscillations by moving of the kink wall. The velocity of the soliton approaches asymptotically an upper-bound value.

II. DRIVEN SINE-GORDON SYSTEM

We will study the equation

$$\Phi_{tt}-c_0^2\Phi_{xx}+\sin\Phi=-f\ , \tag{1}$$

which for $|f|\le 1$ has a multistable ground state

$$\Phi_0=-\arcsin f+2\pi n,\quad n=0,\pm1,\ldots\ . \tag{2}$$

Linear perturbations of Φ_0 are the plane waves (phonons)

$$\psi(x,t)=\psi_0\sum_k \exp[i(kx-\omega_k t)]\ ,\qquad \omega_k^2=(1-f^2)^{1/2}+c_0^2k^2\ . \tag{3}$$

Evidently, the sine-Gordon gap $\Delta(f)=2(1-f^2)^{1/4}$ is lowered by the field.

Our aim is to generalize the perturbation scheme of Eq. (1), so that in the equation for fluctuations $\psi(x,t)$ defined by

$$\Phi(x,t)=\Phi_0+\psi(x,t) \tag{4}$$

together with (1), namely

$$\psi_{tt}-c_0^2\psi_{xx}+(1-f^2)^{1/2}\sin\psi+f(1-\cos\psi)=0\ , \tag{5}$$

we shall go beyond the linear approximation (3) up to third order, so that we get

$$\psi_{tt}-c_0^2\psi_{xx}+\frac{f}{2}\psi^2+(1-f^2)^{1/2}(\psi-\tfrac{1}{6}\psi^3)=0\ . \tag{6}$$

Equation (6) can be transformed to the "normal form"

$$\bar\psi_{tt}-c_0^2\bar\psi_{xx}+A\bar\psi-B\bar\psi^3+F=0\ , \tag{7}$$

where

$$\bar\psi(x,t)=\psi(x,t)+\tan\Phi_0\ , \tag{8}$$

$$A=\frac{1-\frac{1}{2}f^2}{(1-f^2)^{1/2}},\quad B=\tfrac{1}{6}(1-f^2)^{1/2}\ ,\qquad F=\frac{(3-2f^2)}{3(1-f^2)}\arcsin f\ . \tag{9}$$

According to Lal [2] there exists a class of traveling oscillations which obey Eq. (7) exactly,

$$\bar\psi=a\frac{n_1+\cos[\omega(x-vt)]}{n_2+\cos[\omega(x-vt)]}\ , \tag{10}$$

where the coefficients a, ω, n_1, n_2 follow from the substitution of the ansatz (10) into Eq. (7). Then, by eliminating F, they can be obtained as

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$$a^2=\frac{An_2^2}{B(n_2^2+2)}, \quad \omega^2=\frac{2A(n_2^2-1)}{(c_0^2-v^2)(n_2^2+2)}, \quad n_1=-\left[n_2-\frac{2}{n_2}\right], \tag{11}$$

where $n_2>1$, $c_0^2>v^2$. Besides, the coefficient a has to satisfy equation

$$-Aa+Ba^3=F \tag{12}$$

from which, together with (11), is determined the coefficient n_2. The traveling velocity of the oscillations v still remains arbitrary.

The case $v^2>c_0^2$ leads to the solution of the form

$$\bar{\psi}=a\frac{n_1+\cosh[\omega(x-vt)]}{n_2+\cosh[\omega(x-vt)]}, \tag{13}$$

which is in contradiction with numerical calculations. As will be seen later from numerical calculation this behavior of perturbation (traveling with velocity $v^2>c_0^2$) is not present and can be excluded from our considerations.

Further, we shall start from the soliton ansatz (we fix $c_0=1$) $\Phi_K=\pm 4\arctan\exp y$, $y=(\xi-\xi_0)/d$; $\xi=x-vt$ and $d=(1-v^2)^{1/2}$ when $f=0$. For small f, $f^2\ll 1$, we shall investigate the change of the dynamics of the soliton as

$$\Phi(x,t)=-\arcsin f+2\pi n+\Phi_K+\psi(x,t), \quad n=0,\pm 1,\ldots . \tag{14}$$

Then, for fluctuations $\psi(x,t)$ we get

$$\psi_{tt}-c_0^2\psi_{xx}+\left[1-\frac{2}{\cosh^2 y}\right][(1-f^2)^{1/2}\sin\psi+f(1-\cos\psi)]=\pm\frac{2\sinh y}{\cosh^2 y}[-1+(1-f^2)^{1/2}\cos\psi+f\sin\psi]-\frac{2f}{\cosh^2 y}. \tag{15}$$

In the thin-wall approximation, for $d\ll 1$, or $\cosh^{-2}y\approx 0$, Eq. (15) becomes identical with (5) and ψ means then the change of the constant part of the kink due to external field f. Further considerations which followed Eq. (5) remain valid for this case.

Let us use this preliminary calculation in consideration of the following situation. For $t<0$ we shall assume the unperturbed sine-Gordon (SG) equation ($f=0$) with the SG-soliton solution traveling with the velocity v_0,

$$\Phi_{K,v_0}=\pm 4\arctan\exp[(x-v_0t)/(1-v_0^2)^{1/2}]. \tag{16}$$

At $t=0$ we switch on (immediately) the external constant force f. Numerical solution of Eq. (1) is given in Fig. 1 for $f=0.4$ for various times. The solution exhibits these features: (i) a kink wall is preserved and generates the traveling oscillations behind it in the opposite direction. These oscillations will be identified with the above given solution of Eq. (6). (ii) The uniform part of the phase Φ oscillates.

The oscillations of the uniform part $z(t)=\Phi(-\infty,t)$ are governed by the equation

$$\ddot{z}+\sin z=-f, \tag{17}$$

which describes motion of a "particle" in the potential $V(z)=1-\cos z+fz$ with the boundary conditions $z(0)=0$ and $\dot{z}(0)=0$. The resulting motion is periodic if $f<f_{crit}$, where $f_{crit}\approx 0.72461$ can be calculated from the condition at which the "particle" escapes from the potential well.

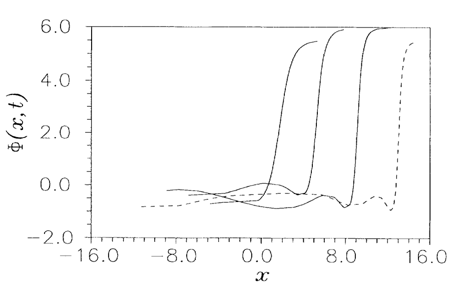

FIG. 1. Time evolution of the SG soliton for $f=0.4$ and $v_0=0$ governed by Eq. (1) with the initial condition (16). From left to right, the moving soliton is depicted at times 4 s, 8 s, 12 s, and 16 s (dashed line).

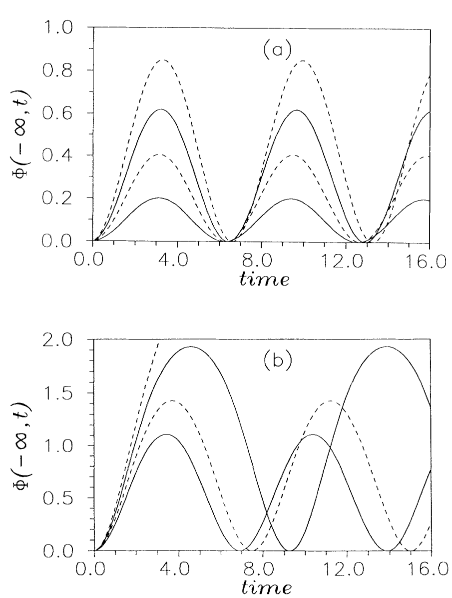

FIG. 2. (a) Solution of Eq. (17). Time evolution of the uniform part $\Phi(-\infty,t)$ of the SG soliton for various values f. According to the increasing amplitude $f=0.1$, $f=0.2$, $f=0.3$, $f=0.4$. (b) Solution of Eq. (17). According to the increasing amplitude $f=0.5$, $f=0.6$, $f=0.7$, $f=0.8>f_{crit}$.

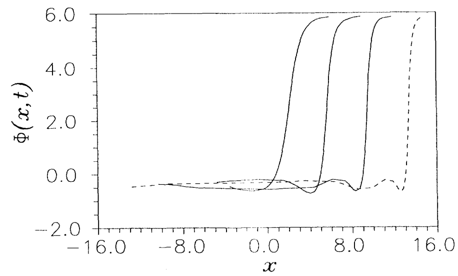

FIG. 3. Time evolution of the SG soliton for $f=0.4$ and $v_0=0$ governed by Eq. (1) with the initial condition (18). From left to right, the moving solition is depicted at times 4 s, 8 s, 12 s, and 16 s (dashed line).

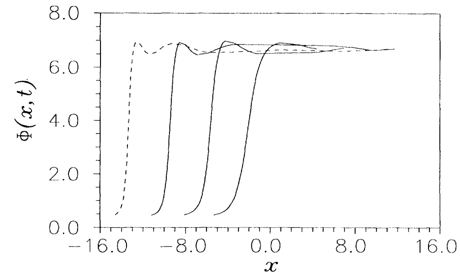

FIG. 7. Time evolution of the SG soliton for $f=-0.4$ and $v_0=0$ governed by Eq. (1) with the initial condition (18). From right to left, the moving solition is depicted at times 4 s, 8 s, 12 s, and 16 s (dashed line).

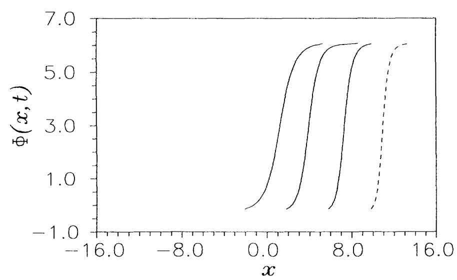

FIG. 4. The same as Fig. 3 for $f=0.2$.

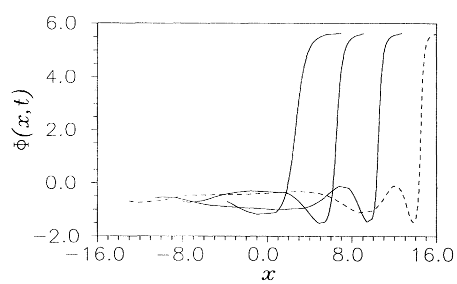

FIG. 5. The same as Fig. 3 for $f=0.6$.

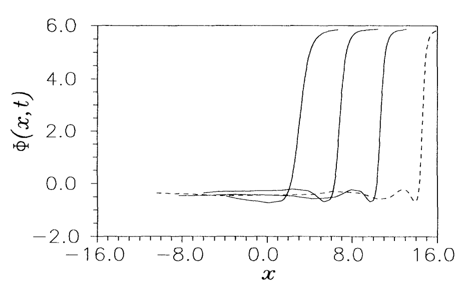

FIG. 6. The same as Fig. 3 for $v_0=0.5c_0$.

Numerical solution of Eq. (1) exhibits the features obtained above analytically. Especially the proportionality of the amplitude of the radiation of the kink to f and the numerical value of $f_{\rm crit}$ are in agreement with the calculations given above. These oscillations of the uniform part (see Fig. 2) can be excluded from the consideration if we switch on external force at $t=0$ immediately as we shift about the stable static solution Φ_0 the whole initial kink Φ_{K,v_0} (16). This means that we solve Eq. (1) by boundary conditions

$$\Phi(x,0)=\Phi_{K,v_0}+\Phi_0 \ . \tag{18}$$

The numerical solutions are given in Figs. 3–7. They show a strong transient nature of the soliton dynamics. Further, comparison of Figs. 3 and 7 shows a symmetry between solutions for given external forces f and $-f$ by initially static kink ($v_0=0$). More precisely, if $f=f_+>0$ and $v_0=0$ with solution $\Phi_+(x,t)=\Phi_0+\psi(x,t)$, then for $f=-f_+<0$ the solution reads $\Phi_-(x,t)=2\pi-\Phi_+(-x,t)$.

III. TRANSIENT DYNAMICS OF THE COLLECTIVE COORDINATES

A convenient insight into the transient dynamics of the driven soliton gives the behavior of collective coordinates, i.e., of the width and the center of the soliton [4,5].

In weak fields f, the dynamics of the kink

$$\Phi(x,t)=\pm 4\arctan\exp\{2[x-X(t)]/L(t)\}+\Phi_s \ , \tag{19}$$

with $\Phi_s=-\arcsin f$ is described by the dynamic equation for

$$p_X(t)=\frac{m_s L_0}{L(t)}\dot{X}(t), \quad p_L(t)=\alpha\frac{m_s L_0}{L(t)}\dot{L}(t) \tag{20}$$

given by [5]

$$\frac{dp_X}{dt}=2\pi f \ ,$$

$$\frac{dp_L}{dt}=-\frac{1}{2m_s L_0}\left[\frac{p_L^2}{\alpha}+p_X^2\right]-\frac{\partial U(L)}{\partial L} \ , \tag{21}$$

where

$$U(L)=\frac{E_s}{2}\left[\frac{L_0}{L(t)}+\cos\Phi_s\frac{L(t)}{L_0}\right],$$
$$m_s=c_0^{-2}E_s,\quad E_s=8c_0,\quad \alpha=\frac{\pi^2}{48}. \tag{22}$$

Initial conditions are

$$L(0)\equiv L_0=2\left[1-\frac{v_0^2}{c_0^2}\right]^{1/2},$$
$$p_X(0)\equiv p_0=m_s v_0\Big/\left[1-\frac{v_0^2}{c_0^2}\right]^{1/2}. \tag{23}$$

Using Eqs. (20)–(23), we get

$$\frac{d}{dt}\left[\frac{\dot L}{L}\right]+\frac{1}{2}\left[\frac{\dot L}{L}\right]^2+\frac{c_0^2}{2\alpha}\left[\frac{\cos\Phi_s}{L_0^2}-\frac{1}{L^2}\right]+2\beta^2\left[t+\frac{p_0}{2\pi f}\right]^2=0, \tag{24}$$

where $\beta=\pi f/\alpha^{1/2}m_sL_0$ and p_0 is an initial momentum $p_X(0)=p_0$. A way to solve Eq. (24) is to use the ansatz $L(t)=g^2(t)$ which gives

$$\ddot g+[\Omega^2+\beta^2(t^2+tp_0/(f\pi))]g-\frac{c_0^2}{4\alpha g^3}=0, \tag{25}$$

where

$$\Omega^2=\frac{c_0^2}{4\alpha L_0^2}\left[\cos\Phi_s+\frac{p_0^2}{m_s^2c_0^2}\right].$$

If $\beta=0$ ($f=0$), then Eq. (24) has an exact solution [4]

$$L(t)=\frac{cL_0}{\left[1+\frac{p_0^2}{m_s^2c_0^2}\right]^{1/2}}[1\pm\sin 2\Omega_0(t-t_1)], \tag{26}$$

where

$$\Omega_0^2=\frac{c_0^2}{4\alpha L_0^2}\left[1+\frac{p_0^2}{m_s^2c_0^2}\right], \tag{27}$$

c, t_1 are constants.

Transient behavior for small $t\ll p_0/\pi f$ can be found approximately by separation of the slowly varying part of g, $\ddot g_s\approx 0$, and the oscillating part g_1, $g=g_s+g_1$ so that

$$L_s=g_s^2=\frac{L_0}{[\cos\Phi_s+(p_0+2\pi f t)^2/(m_s^2c_0^2)]^{1/2}}. \tag{28}$$

The oscillating part $g_1\ll g_s$ then fulfills

$$\ddot g_1+\Omega^2 g_1+\frac{3c_0^2}{4\alpha g_s^4}g_1=0 \tag{29}$$

or

$$\ddot g_1+\left[\Lambda^2+4\beta^2\frac{p_0t}{\pi f}\right]g_1=0, \tag{30}$$

where

$$\Lambda^2=\frac{c_0^2}{\alpha L_0^2}\left[\cos\Phi_s+\frac{p_0^2}{m_s^2c_0^2}\right]. \tag{31}$$

If we introduce a new variable

$$\xi\equiv(\Lambda^2+\gamma t)/\gamma^{2/3},\quad \gamma=4\beta^2\frac{p_0}{\pi f}, \tag{32}$$

then Eq. (30) turns to $[g_1(t)\equiv\eta(\xi)]$

$$\frac{d^2\eta}{d\xi^2}+\xi\eta=0. \tag{33}$$

The solution to Eq. (33) is a cylindric function [6]

$$\eta(\xi)=\xi^{1/2}Z_{1/3}(2\xi^{3/2}/3), \tag{34}$$

where

$$Z_\nu(x)=c_1J_\nu(x)+c_2Y_\nu(x),\quad x=\tfrac{2}{3}\xi^{3/2}, \tag{35}$$

and J_ν, Y_ν are Bessel functions of the first order,

$$Y_\nu(x)=\frac{\cos(\nu\pi)J_\nu(x)-J_{-\nu}(x)}{\sin(\nu\pi)}, \tag{36}$$

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left[\frac{x}{2}\right]^{\nu+2k}. \tag{37}$$

For $|f|\ll 1$ ($x\gg 1$), $\nu=\frac{1}{3}$, we have

$$J_{1/3}(x)\approx\left[\frac{2}{\pi x}\right]^{1/2}\cos\left[x-\frac{5\pi}{12}\right],$$
$$J_{-1/3}(x)\approx\left[\frac{2}{\pi x}\right]^{1/2}\cos\left[x-\frac{\pi}{12}\right]. \tag{38}$$

Evidently, the transient oscillations of the collective coordinates $L(t)$ and $X(t)$ given by $g_1(t)\equiv\eta(\xi)$ are induced by the external field f. For small f and small t we get finally

$$g_1(t)=\left[\frac{3}{\pi}\right]^{1/2}\gamma^{1/6}(\Lambda^2+\gamma t)^{-1/4}\left[\left[c_1+\frac{c_2}{\sqrt{3}}\right]\cos\left[\frac{2(\Lambda^2+\gamma t)^{3/2}}{3\gamma}-\frac{5\pi}{12}\right]-\frac{2}{\sqrt{3}}c_2\cos\left[\frac{2(\Lambda^2+\gamma t)^{3/2}}{3\gamma}-\frac{\pi}{12}\right]\right]. \tag{39}$$

IV. CONCLUSIONS

We have shown that a sine-Gordon soliton driven by a small constant external field $f<f_{\rm crit}=0.7246$ exhibits periodic nonsinusoidal traveling oscillations. These oscillations can be found analytically in the limit of a vanishing soliton width (truncation approximation for a soliton and fluctuation dynamics) as an exact solution of a non-

linear equation for fluctuations. The radiation appears as a result of an interplay of the nonlinearity and of a constant field. According to the numerical results the kink can be accelerated up to the maximum velocity $v_{\max} < c_0$, so that it appears that the energy obtained from the field transforms to the radiation. Numerical investigations show the highly transient nature of the dynamics. An analytical treatment is possible for small times for the dynamics of the collective coordinates. In the transient region the soliton velocity consists of two parts, one growing approximately linearly with the time, $\dot{X}_s(t) = (2\pi f t / m_s L_0) L_s(t)$, and an oscillating part, $v_{\rm osc} \approx (4\pi f t / m_s L_0)(L_s)^{1/2} g_1(t), g_1$ given by (39).

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